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ON THE BOUNDARIES OF THE PARAMETRIC RESONANCE DOMAIN†

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The problem of stability for a system of linear differential equations with coefficients which are periodic in time and depend on the parameters is considered. The singularities of the general position arising at the boundaries of the stability and instability (parametric resonance) domains in the case of two and three parameters are listed. A constructive approach is proposed which enables one, in the first approximation, to determine the stability domain in the neighbourhood of a point on the boundary (regular or singular) from the information at this point. This approach enables one to eliminate a tedious numerical analysis of the stability region in the neighbourhood of the boundary point and can be employed to construct the boundaries of parametric resonance domains. As an example, the problem of the stability of the oscillations of an articulated pipe conveying fluid with a pulsating velocity is considered. In the space of three parameters (the average fluid velocity and the amplitude and frequency of pulsations) a singularity of the boundary of the stability domain of the "dihedral angle" type is obtained and the tangential cone to the stability domain is calculated. © 2001 Elsevier Science Ltd. All rights reserved.

The main problem of the theory of parametric resonance is to construct stability and instability domains in the parameter space [1]. It is well known that the boundary of the stability domain in the parameter plane can have singularities, for example, a point of inflection. When there are a large number of parameters in the system one must expect more complex singularities to arise, the occurrence of which may be reflected in the physical properties of the system. Of all the types of singularities that may occur, it is most important to investigate singularities of the general position (typical singularities).

Despite the advances in modern computers, numerical multiparameter analysis of the stability of periodic systems is a complicated problem, in view of the need to carry out a multiple integration of differential equations. This applies particularly to systems with a large number of degrees of freedom and when analysing stability domains close to singular points of the boundary.

Below we present a classification of singularities of the general position of the boundaries of stability domains for systems of linear differential equations of general type with coefficients which are periodic in time and depend on two or three parameters. A constructive approach is proposed which enables one, in the first approximation, to determine the stability domain in the neighbourhood of a point on its boundary using only information at this point: the values of multipliers, eigenvectors and associated vectors of the monodromy matrix and the first derivatives of the system operator with respect to parameters. The proposed approach uses general formulae for the derivatives of the monodromy matrix with respect to parameters [2] and perturbation theory of eigenvalues of the matrices [3, 4]. Unlike a number of previous investigations [1] no assumption regarding closeness of the periodic system to an autonomous system is used.

The singularities of boundaries of the stability domains of autonomous systems were investigated previously in [5–8].

1. THE MONODROMY MATRIX AND ITS DERIVATIVES

Consider a system of linear homogeneous differential equations with periodic coefficients

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x} \tag{1.1}$$

where $\mathbf{x}$ is a real vector of dimension m and $\mathbf{G} = \mathbf{G}(t)$ is a real matrix function of dimension $m \times m$, continuous in time t and periodic with minimum period T, $\mathbf{G}(t+T) = \mathbf{G}(t)$.

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Suppose the vector function $\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)$ are the solutions of system of equations (1.1), which are linearly independent for a certain $t = t_0$ (and consequently for all $t \in R$ also). The matrix $\mathbf{X}(t)$, composed of the column vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_m(t)$, is called the fundamental matrix. If, in addition, $\mathbf{X}(t)$ satisfies the initial condition $\mathbf{X}(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, then $\mathbf{X}(t)$ is called the matriciant [1]. Note that the fundamental matrix $\mathbf{X}(t)$ is non-singular for all $t \in R$.

If $\mathbf{X}(t)$ is the matriciant of system of equations (1.1), the matrix

$$\mathbf{F} = \mathbf{X}(T) \tag{1.2}$$

is called the transition matrix or the monodromy matrix [1]. The monodromy matrix $\mathbf{F}$ can be obtained by m-fold numerical integration of system of equations (1.1) over the period T with initial conditions which are the columns of the identity matrix $\mathbf{I}$.

The stability of system (1.1) is determined by the following conditions, imposed on eigenvalues (multipliers) of the monodromy matrix $\mathbf{F}$ [1]: if all the multipliers lie inside the unit circle $|\rho_j| < 1$ ($j = 1, \ldots, m$), system (1.1) is asymptotically stable; if at least one multiplier lies outside this circle, system (1.1) is unstable.

We will assume that the matrix $\mathbf{G}$ in (1.1) depends not only on time but depends continuously on the vector of the real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. Together with system (1.1) we will also consider the adjoint system [1]

$$\dot{\mathbf{y}} = -\mathbf{G}^T\mathbf{y} \tag{1.3}$$

where $\mathbf{y}$ is a real vector of dimension m. The equations for the matriciants of system (1.1) and (1.3) have the form [1]

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}; \quad \dot{\mathbf{Y}} = -\mathbf{G}^T\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{I} \tag{1.4}$$

Matriciants $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are connected by the relation $\mathbf{X}(t)^T\mathbf{Y}(t) = \mathbf{I}$ [1].

Explicit expressions for the derivatives of the monodromy matrix $\mathbf{F}$ with respect to parameters were obtained in [2] in terms of the matriciants $\mathbf{X}(t)$ and $\mathbf{Y}(t)$. The first and second derivatives have the form

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial p_k} &= \mathbf{F}\int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_k}\mathbf{X}\,dt \\
\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} &= \mathbf{F}\left[\int_0^T \mathbf{Y}^T \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j}\mathbf{X}\,dt + \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X}\left(\int_0^\tau \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\,d\xi\right)d\tau + \int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\left(\int_0^\tau \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_i}\mathbf{X}\,d\xi\right)d\tau\right]
\end{aligned} \tag{1.5}$$

We will derive an expression for the derivatives of the monodromy matrix of arbitrary order. To do this we introduce the vector $\mathbf{h} = (h^1, \ldots, h^n)$ with non-negative integer components, not equal to zero simultaneously, and we will use the notation $|\mathbf{h}| = h^1 + \ldots + h^n$, $\mathbf{h}! = h^1! \ldots h^n!$. Then, using the functions

$$\mathbf{B}_\mathbf{h}(t) = \frac{1}{\mathbf{h}!}\mathbf{Y}^T(t)\frac{\partial^{|\mathbf{h}|}\mathbf{G}}{\partial p_1^{h^1} \ldots \partial p_n^{h^n}}\mathbf{X}(t)$$

we write an expression for the derivative of order $|\mathbf{h}|$ in the form

$$\frac{\partial^{|\mathbf{h}|}\mathbf{F}}{\partial p_1^{h^1} \ldots \partial p_n^{h^n}} = \mathbf{h}!\mathbf{F}_0 \sum_{\substack{\mathbf{h}_1 + \ldots + \mathbf{h}_s = \mathbf{h},\\ s = 1, \ldots, |\mathbf{h}|}} \int_0^T \mathbf{B}_{\mathbf{h}_1}(t_1)\left(\int_0^{t_1} \mathbf{B}_{\mathbf{h}_2}(t_2)\left(\ldots \int_0^{t_{s-1}} \mathbf{B}_{\mathbf{h}_s}(t_s)dt_s \ldots\right)dt_2\right)dt_1$$

The formulae for the derivatives of the monodromy matrix can be extended to the case when the period is a continuous function of the vector of the parameters $T = T(\mathbf{p})$. For example, differentiating the monodromy matrix $\mathbf{F}(\mathbf{p}) = \mathbf{X}(T(\mathbf{p}), \mathbf{p})$ with respect to p_k and using Eqs (1.4), we have

$$\frac{\partial \mathbf{F}}{\partial p_k} = \frac{\partial \mathbf{X}(T(\mathbf{p}), \mathbf{p})}{\partial p_k} = \left(\frac{\partial \mathbf{X}}{\partial p_k}\right)_T + \left(\frac{\partial \mathbf{X}}{\partial t}\right)_T \frac{\partial T}{\partial p_k} = \left(\frac{\partial \mathbf{X}}{\partial p_k}\right)_T + \mathbf{G}(T)\mathbf{F}\frac{\partial T}{\partial p_k}$$

The first term on the right-hand side is determined by the first expression of (1.5). As a result we obtain a formula for the first derivative of the monodromy matrix for the case when the period depends on parameters

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F}\int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_k}\mathbf{X}dt + \mathbf{G}(T)\mathbf{F}\frac{\partial T}{\partial p_k} \tag{1.6}$$

2. THE BOUNDARY OF THE PARAMETRIC RESONANCE DOMAIN AND ITS SINGULARITIES

Consider the monodromy matrix $\mathbf{F}(\mathbf{p})$ of periodic system (1.1). The matrix function $\mathbf{F}(\mathbf{p})$ is continuous (the term "a family of matrices" is also used in the mathematical literature). The condition for asymptotic stability divides the space of parameters R^n into stability and instability (parametric resonance) domains. The transition from a stability domain to an instability domain is accompanied by certain multipliers leaving the unit circle. In particular, the passage of the real multipliers through the points 1 and –1 is called the fundamental resonance, while the exit of a complex-conjugate pair through the unit circle at points $\exp(\pm i\omega)(\omega \neq \pi k,\ k \in Z)$ is called combination resonance.

In the general case, the boundary of the stability domain is a continuous hypersurface in parameter space having singularities (points where continuity is lost). In this case, we are primarily interested in singularities of the general position (typical singularities). The occurrence of such singularities can always be expected when investigating specific systems. As regards singularities of non-general position, they are a consequence of a certain degeneracy or symmetry of the system and disappear for a movement of the family as small as desired [5]. Below we will investigate the singularities of the general position of the boundary of the parametric resonance domain, since they are the most important from the point of view of applications, although the methods developed can also be used to investigate singularities of non-general position.

We will denote the types of the boundary points [5] by the product of multipliers which are on the unit circle with powers equal to the dimensions of the corresponding Jordan blocks. For example, $1^2\exp(\pm i\omega_1)\exp(\pm i\omega_2)$ denotes the presence in the monodromy matrix $\mathbf{F}$ of a double $\rho = 1$ *with second-order Jordan block and two pairs of simple multipliers* $\rho = \exp(\pm i\omega_1), \exp(\pm i\omega_2)$ such that $\omega_1, \omega_2 \in (0, \pi)$, $\omega_1 \neq \omega_2$. For convenience we will introduce certain types of short notation as follows:

$$B_1(1),\quad B_2(-1),\quad B_3(\exp(\pm i\omega)) \tag{2.1}$$

$$C_1(1^2),\quad C_2((-1)^2) \tag{2.2}$$

$$D_1(1^3),\quad D_2((-1)^3),\quad D_3((\exp(\pm i\omega))^2) \tag{2.3}$$

and also combinations of them

$$B_{12}(1(-1)),\quad B_{13}(1\exp(\pm i\omega)),\quad B_{23}((-1)\exp(\pm i\omega)),\quad B_{33}(\exp(\pm i\omega_1)\exp(\pm i\omega_2)) \tag{2.4}$$

$$B_{123}(1(-1)\exp(\pm i\omega))$$

$$\begin{gathered} B_{133}(1\exp(1\pm i\omega_1)\exp(\pm i\omega_2)),\quad B_{233}((-1)\exp(\pm i\omega_1)\exp(\pm i\omega_2)) \\ B_{333}(\exp(\pm i\omega_1)\exp(\pm i\omega_2)\exp(\pm i\omega_3)),\quad C_1B_2(1^2(-1)) \\ C_2B_1((-1)^2 1),\quad C_1B_3(1^2\exp(\pm i\omega)),\quad C_2B_3((-1)^2\exp(\pm i\omega)) \end{gathered} \tag{2.5}$$

For example, $B_1(1^2)$ denotes the presence of a double multiplier $\rho = 1$ with the second-order Jordan block, while the combination C_1B_3 at a boundary point denotes the presence on the unit circle of a double multiplier $\rho = 1$ and a complex-conjugate pair $\rho = \exp(\pm i\omega)$.

For our further investigation we will use the results obtained in [5, 9], which enable us to determine the codimension of a defined type of set of boundary points. The codimension is the difference in the dimension of the whole space and the dimension of the given set. Obviously, an n-parametric family one only encounters types to boundary points of codimension no higher than n. Below we will consider the case when $n = 1, 2, 3$. It follows from the results obtained previously in [5, 9] that in the case of the general position all types of boundary points of codimension 1 are covered by list (2.1), those of codimension 2 are covered by (2.2) and (2.4), while those of codimension 3 are covered by (2.3) and (2.5).

A qualitative analysis of the stability domain in the neighbourhood of points of its boundary was carried out using the theory of versal deformations [5, 9]. Without going into the details of this theory, we will proceed directly to its consequence as far as boundary points of the type (2.1)–(2.5) are concerned (for more detail see [5, 10], where autonomous systems are investigated). The stability domain in the neighbourhood of boundary points (2.1)–(2.3), apart from a continuous replacement of the coordinates, is equivalent to the stability domains of the matrix $\mathbf{F}'(\mathbf{p}')$, $\mathbf{p}' = (p'_1, \dots, p'_d)$ in the neighbourhood of $\mathbf{p}' = 0$, which, in each specific case, has the form

$$B_1 : (1 + p_1'), \quad B_2 : (-1 + p_1'), \quad B_3 : (\exp(p_1' + i\omega))$$

$$C_1 : \begin{Vmatrix} 1 & 1 \\ p_2' & 1 + p_1' \end{Vmatrix}, \quad C_2 : \begin{Vmatrix} -1 & 1 \\ p_2' & -1 + p_1' \end{Vmatrix}$$

$$D_1 : \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ p_3' & p_2' & 1 + p_1' \end{Vmatrix}, \quad D_2 : \begin{Vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ p_3' & p_2' & -1 + p_1' \end{Vmatrix} \tag{2.6}$$

$$D_3 : \begin{Vmatrix} \exp(p_1' + i\omega) & 1 \\ p_2' + ip_3' & \exp(p_1' + i\omega) \end{Vmatrix}$$

The matrices in (2.6) are responsible for the stability of the blocks of miniversal deformations (normal forms), derived in [5, 9]. The corresponding matrices for combined types (2.4) and (2.5) have a block-diagonal form and consist of blocks (2.6), where the parameters in different blocks are assumed to be independent. For example, the matrices $\mathbf{F}'(\mathbf{p}')$ corresponding to a type C_1B_2 boundary point (combinations of C_1 and B_3), have the form

$$\begin{Vmatrix} 1 & 1 & 0 \\ p_2' & 1 + p_1' & 0 \\ 0 & 0 & \exp(p_3' + i\omega) \end{Vmatrix}$$

It is easy (although quite lengthy) to investigate the stability of matrices $\mathbf{F}'(\mathbf{p}')$ for cases (2.1)–(2.5). As a results, the form of the stability domain (apart from a continuous replacement of the parameters) and also the type and form of the adjoining boundary can be determined locally.

As an example, consider the case C_1. The roots of the characteristic equation of matrix $\mathbf{F}'(\mathbf{p}')$ (2.6) have the form

$$\rho = 1 + p_1'/2 \pm \sqrt{p_2' + (p_1')^2/4}$$

Introducing the non-degenerate replacement of parameters $q_1 = p'_1/2$, $q_2 = p'_2 + (p'_1)^2/4$, we obtain an expression for the maximum (in absolute value) multiplier

$$\max|\rho|^2 = \begin{cases} 1 + 2q_1 + q_1^2 - q_2, & q_2 < 0 \\ (1 + q_1 + \sqrt{q_2})^2, & q_2 \geqslant 0 \end{cases}$$

The stability domain, defined by the condition $\max|\rho| < 1$, is shown in Fig. 1. The boundary of the stability domain has an inflection at the point C_1 and consists of curves of the type B_1 and B_3, which intersect at a non-zero angle, corresponding to the boundaries of the fundamental and combination resonances.

Other types of boundary points can be investigated in a similar way. We summarize the results obtained in the form of a theorem.

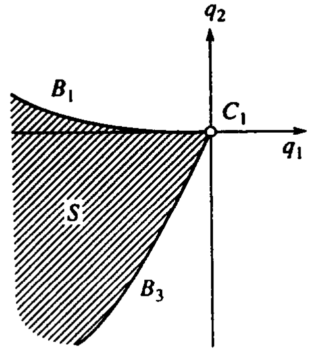

Fig. 1

Theorem 1. In the case of the general position, the boundary of the stability domain of system (1.1) consists of the following:

(a) in the case of a single parameter, it consists of isolated points of the types B_1, B_2 and B_3 (2.1), corresponding to the fundamental and combination resonances;

(b) in the case of two parameters it consists of continuous curves of type (2.1), which intersect transversally (at a non-zero angle) at points of inflection of types (2.2) and (2.4),

(c) in the case of three parameters it consists of smooth surfaces of type (2.1), the singularities of which are curves of types (2.2) and (2.4) – a "dihedral angle" and isolated points of type (2.3): D_1 and D_2 – a "break of an edge", D_3 – a "deadlock of an edge", and (2.5) – a "trihedral angle".

The stability domains in the region of singular points of the above types, apart from a non-degenerate continuous replacement of parameters (a diffeomorphism), have the form shown in Figs 2 and 3 (the stability domain is represented by the letter S).

Remark 1. The boundary of the stability domain in the case D_3 is a diffeomorphic to the surface specified by the equation $xy^2 = z^2, x \geqslant 0, y \geqslant 0$ and is a part of the so-called "Whitney–Kelly umbrella" [5]. In the cases D_1 and D_2 the boundary of the stability domain is diffeomorphic to the surface $x^2y^2 = z^2, x \geqslant 0, y \geqslant 0$.

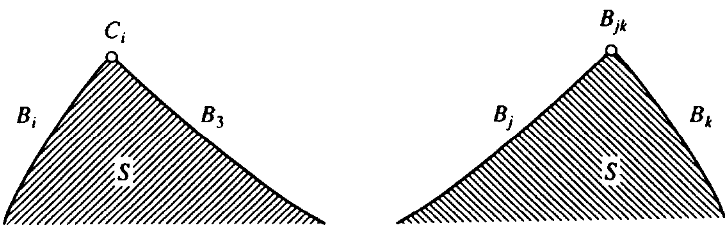

Fig. 2

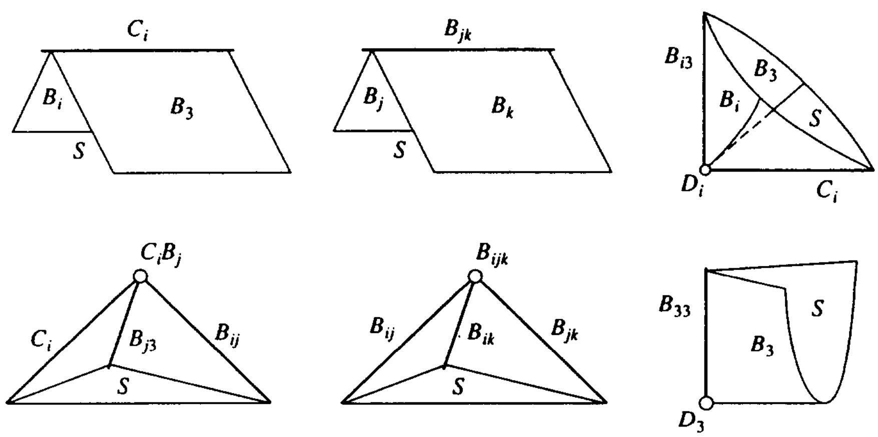

Fig. 3

Remark 2. The singularities of the boundary of the stability domain in the case of the general position for periodic systems (1.1), which depend on two or three parameters, are equivalent, apart from a diffeomorphism, to the singularities for autonomous systems, listed earlier in [5], despite the difference in the stability criteria and the types of boundary points. Note, however, that the types of singularities B_{12}, B_{123}, C_1B_2 and C_2B_1 for periodic systems are essentially new compared with autonomous systems. The point is that, using Lyapunov's theorem on reducibility [1] of system (1.1), it reduces to a system with constant coefficients. Here both multipliers $\rho = 1$ and $\rho = -1$ convert into the zero eigenvalue of the matrix of an autonomous system forming two Jordan blocks, which is the case of non-general position for autonomous systems.

3. BIRFURCATION OF MULTIPLIERS

We will consider the eigenvalue problem for the monodromy matrix (1.2)

$$\mathbf{Fu} = \rho\mathbf{u} \tag{3.1}$$

where $\mathbf{u}$ is the eigenvector of dimension m, corresponding to the multiplier ρ. It is of interest to calculate the change of the multipliers depending on the change in the parameters $p_1, \dots, p_n$. For this purpose we consider, in the parameter space, the continuous single-parameter curves $\mathbf{p}(\varepsilon)$, passing through the points $\mathbf{p}_0 = \mathbf{p}(0)$, where ε is a small parameter. We determine the vectors

$$\mathbf{e} = (d\mathbf{p}/d\varepsilon)_{\varepsilon=0}, \quad \mathbf{d} = \frac{1}{2}(d^2\mathbf{p}/d\varepsilon^2)_{\varepsilon=0}$$

The increment of the parameter vector can then be represented in the form

$$\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d} + o(\varepsilon^2)$$

As a result of a perturbation of the vector $\mathbf{p}_0$ of the monodromy matrix $\mathbf{F}$ we obtain an increment, which we represent in the form of a series

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}_0 + \varepsilon\mathbf{F}_1 + \varepsilon^2(\mathbf{F}_2 + \mathbf{F}_d) + \dots \tag{3.2}$$

The matrices $\mathbf{F}_i (i = 0, 1, 2)$ and $\mathbf{F}_d$ are defined by the relations

$$\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0), \quad \mathbf{F}_1 = \sum_{j=1}^n \frac{\partial\mathbf{F}}{\partial p_j}e_j, \quad \mathbf{F}_2 = \frac{1}{2}\sum_{s,k=1}^n \frac{\partial^2\mathbf{F}}{\partial p_s \partial p_k}e_s e_k, \quad \mathbf{F}_d = \sum_{j=1}^n \frac{\partial\mathbf{F}}{\partial p_j}d_j \tag{3.3}$$

The derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$.

Suppose ρ_0 is the multiplier of matrix $\mathbf{F}_0$. As a result of a perturbation of the vector of parameters $\mathbf{p}_0$, the multiplier ρ_0 takes an increment. According to the perturbation theory of non-self-adjoint operators [3, 4], this increment has different representations depending on the Jordan structure of the matrix $\mathbf{F}_0$.

1. In the case of a simple eigenvalue ρ_0 the increment of a multiplier can be represented in the form of an expansion in integer powers of ε

$$\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \dots \tag{3.4}$$

where the first term is

$$\rho_1 = \mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0 / \mathbf{v}_0^T\mathbf{u}_0 \tag{3.5}$$

The vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ are the right and left eigenvectors, corresponding to the multiplier ρ_0

$$\mathbf{F}_0\mathbf{u}_0 = \rho_0\mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{F}_0 = \rho_0\mathbf{v}_0^T \tag{3.6}$$

Introducing the real vectors $\mathbf{r}$ and $\mathbf{k}$ of dimension n with components

$$r^s + ik^s = \mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_0 / \mathbf{v}_0^T\mathbf{u}_0, \quad s = 1,\dots, n \tag{3.7}$$

where i is the square root of -1, we can write expansion (3.4), taking relations (3.5) and (3.7) into account, in the form [4, 6]

$$\rho = \rho_0 + [(\mathbf{r},\mathbf{e}) + i(\mathbf{k},\mathbf{e})]\varepsilon + o(\varepsilon) \tag{3.8}$$

The scalar product in R^n is denoted by brackets. If ρ_0 is a real number, it follows from (3.7) that $\mathbf{k} = 0$.

2. Consider the case of a double multiplier ρ_0 with second-order Jordan block. This means that, at $\mathbf{p} = \mathbf{p}_0$, the eigenvector and adjoint vector $\mathbf{u}_0$ and $\mathbf{u}_1$, defined by the

$$\mathbf{F}_0\mathbf{u}_0 = \rho_0\mathbf{u}_0, \quad \mathbf{F}_0\mathbf{u}_1 = \rho_0\mathbf{u}_1 + \mathbf{u}_0 \tag{3.9}$$

equations correspond to the multiplier ρ_0. For the left eigenvector and adjoint vector we have, respectively

$$\mathbf{v}_0^T\mathbf{F}_0 = \rho_0\mathbf{v}_0^T, \quad \mathbf{v}_1^T\mathbf{F}_0 = \rho_0\mathbf{v}_1^T + \mathbf{v}_0^T \tag{3.10}$$

Assuming the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ to be fixed, we introduce the normalization

$$\mathbf{v}_0^T\mathbf{u}_1 = 1, \quad \mathbf{v}_1^T\mathbf{u}_1 = 0 \tag{3.11}$$

which uniquely defines the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$.

As a consequence of the perturbation of the parameter vector, an increment of the multiplier can be represented in the form of a series [3]

$$\rho = \rho_0 + \varepsilon^{1/2}\rho_1 + \varepsilon\rho_2 + \varepsilon^{3/2}\rho_3 + \ldots \tag{3.12}$$

The first coefficients ρ_1 and ρ_2 have the form [4, 6]

$$\rho_1 = \pm\sqrt{\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0}, \quad \rho_2 = (\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{F}_1\mathbf{u}_0)/2 \tag{3.13}$$

We introduce real vectors $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$ with the components defined by the relations

$$\begin{aligned} f_1^s + if_2^s &= \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_0 \\ q_1^s + iq_2^s &= \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_1^T \frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_0, \quad s = 1, \ldots, n \end{aligned} \tag{3.14}$$

Taking relations (3.13) and (3.14) into account, we can write expansion (3.12) in the form

$$\rho = \rho_0 \pm \sqrt{\varepsilon[(\mathbf{f}_1,\mathbf{e}) + i(\mathbf{f}_2,\mathbf{e})]} + \varepsilon[(\mathbf{q}_1,\mathbf{e}) + i(\mathbf{q}_2,\mathbf{e})]/2 + o(\varepsilon) \tag{3.15}$$

Relation (3.15) describes splitting of the double multiplier ρ_0 with one eigenvector $\mathbf{u}_0$ in the non-degenerate case

$$\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0 = (\mathbf{f}_1,\mathbf{e}) + i(\mathbf{f}_2,\mathbf{e}) \neq 0$$

If ρ_0 is a real number, we have $\mathbf{f}_2 = \mathbf{q}_2 = 0$.

3. Consider the case of a triple real multiplier ρ_2, corresponding to a third-order Jordan block. The Jordan blocks for the right and left eigenvectors and adjoint vectors have the form

$$\begin{aligned} &\mathbf{F}_0\mathbf{u}_0 = \rho_0\mathbf{u}_0, \quad \mathbf{F}_0\mathbf{u}_1 = \rho_0\mathbf{u}_1 + \mathbf{u}_0, \quad \mathbf{F}_0\mathbf{u}_2 = \rho_0\mathbf{u}_2 + \mathbf{u}_1 \\ &\mathbf{v}_0^T\mathbf{F}_0 = \rho_0\mathbf{v}_0^T, \quad \mathbf{v}_1^T\mathbf{F}_0 = \rho_0\mathbf{v}_1^T + \mathbf{v}_0^T, \quad \mathbf{v}_2^T\mathbf{F}_0 = \rho_0\mathbf{v}_2^T + \mathbf{v}_1^T \end{aligned} \tag{3.16}$$

Assuming the vectors $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ to be fixed, we impose the following normalization conditions on $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{v}_0^T\mathbf{u}_2 = 1, \quad \mathbf{v}_1^T\mathbf{u}_2 = 0, \quad \mathbf{v}_2^T\mathbf{u}_2 = 0 \tag{3.17}$$

which uniquely define these vectors. In view of the fact that ρ_0 is a real number, the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ can be chosen to be real.

A variation of the vector of parameters $\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d} + o(\varepsilon^2)$ leads, in general, to splitting of the triple multiplier into three simple complex multipliers. This bifurcation is described by the relation [3]

$$\rho = \rho_0 + \varepsilon^{1/3}\rho_1 + \varepsilon^{2/3}\rho_2 + \varepsilon\rho_3 + \dots \tag{3.18}$$

where $\rho_1 = (\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0)^{1/3}$, and the three complex values of the root correspond to three different ρ. We introduce the vector $\mathbf{g}$ with components

$$g^s = \mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_0, \quad s = 1,\dots, n \tag{3.19}$$

Bifurcation (3.18) then takes the form

$$\rho = \rho_0 + (\varepsilon(\mathbf{g},\mathbf{e}))^{1/3} + o(\varepsilon^{1/3}) \tag{3.20}$$

We now consider the degenerate case

$$\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0 = (\mathbf{g},\mathbf{e}) = 0$$

The triple multiplier ρ_0 in this case, as before, can split into three simple multipliers, but in this case two of the multipliers are expanded in powers of $\varepsilon^{1/2}$

$$\rho = \rho_0 + \varepsilon^{1/2}\nu_1 + \varepsilon\nu_2 + \dots \tag{3.21}$$

while the third multiplier is expanded in powers of ε [3]

$$\rho = \rho_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots \tag{3.22}$$

Using expansions (3.21) and (3.22), from the equations of the perturbation method we obtain expressions for the first coefficients

$$\begin{aligned}
\mu_1 &= \frac{\mathbf{v}_0^T\mathbf{F}_1\mathbf{G}_0(\mathbf{F}_1\mathbf{u}_0) - \mathbf{v}_0^T(\mathbf{F}_2 + \mathbf{F}_d)\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{F}_1\mathbf{u}_0} \\
\nu_1^2 &= \mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{F}_1\mathbf{u}_0 \\
\nu_2 &= [-\mu_1 + \mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_2 + \mathbf{v}_1^T\mathbf{F}_1\mathbf{u}_1 + \mathbf{v}_2^T\mathbf{F}_1\mathbf{u}_0]/2
\end{aligned} \tag{3.23}$$

where $\mathbf{G}_0$ is an operator inverse to $\mathbf{F}_0 - \rho_0\mathbf{I}$ ($\mathbf{I}$ is the identity matrix). Since the matrix $\mathbf{F}_0 - \rho_0\mathbf{I}$ is degenerate, the operator $\mathbf{G}_0$ is defined on a set of vectors $\mathbf{w}$, which satisfy the orthogonality condition $\mathbf{v}_0^T\mathbf{w} = 0$. The quantity $\mathbf{G}_0(\mathbf{w})$ is defined, apart from an additive term $\gamma\mathbf{u}_0$, $\gamma = \text{const}$, on which μ_1 does not depend, since $\mathbf{v}_0^T\mathbf{F}_1\mathbf{u}_0 = 0$. The operator $\mathbf{G}_0(\mathbf{w})$ can be represented in the form $\mathbf{G}_0(\mathbf{w}) = \mathbf{A}_0^{-1}\mathbf{w} + \gamma\mathbf{u}_0$, where $\mathbf{A}_0 = \mathbf{F}_0 - \rho_0\mathbf{I} - \mathbf{v}_0\mathbf{v}_2^T$ is a non-degenerate matrix [1].

We introduce real vectors $\mathbf{h}$ and $\mathbf{t}$ of dimension n with the components

$$\begin{aligned}
h^s &= \mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_1^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_0 \\
t^s &= \mathbf{v}_0^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_2 + \mathbf{v}_1^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_2^T\frac{\partial\mathbf{F}}{\partial p_s}\mathbf{u}_0, \quad s = 1,\dots,n
\end{aligned} \tag{3.24}$$

and a real matrix $\mathbf{R}$ of dimension $n \times n$, defined by the relation

$$(\mathbf{Re},\mathbf{e}) = \mathbf{v}_0^T\mathbf{F}_1\mathbf{G}_0(\mathbf{F}_1\mathbf{u}_0) - \mathbf{v}_0^T\mathbf{F}_2\mathbf{u}_0$$

$$\mathbf{R} = \left[\mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_i} [\mathbf{F}_0 - \rho_0 \mathbf{I} - \mathbf{v}_0 \mathbf{v}_2^T]^{-1} \frac{\partial \mathbf{F}}{\partial p_j} \mathbf{u}_0 - \mathbf{v}_0^T \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} \mathbf{u}_0 \right], \quad i,j = 1,\ldots,n \tag{3.25}$$

Using relations (3.23)–(3.25) we can represent bifurcation (3.21), (3.22) in the form

$$\rho = \rho_0 \pm \sqrt{\varepsilon(\mathbf{h},\mathbf{e})} + \left[(\mathbf{t},\mathbf{e}) - \frac{(\mathbf{Re},\mathbf{e}) - (\mathbf{g},\mathbf{d})}{(\mathbf{h},\mathbf{e})} \right] \frac{\varepsilon}{2} + o(\varepsilon)$$

$$\rho = \rho_0 + \frac{(\mathbf{Re},\mathbf{e}) - (\mathbf{g},\mathbf{d})}{(\mathbf{h},\mathbf{e})} \varepsilon + o(\varepsilon) \tag{3.26}$$

The vector $\mathbf{g}$ is defined in (3.19).

Expression (3.26) describes the bifurcation of the triple multiplier ρ_0 in the degenerate case $(\mathbf{g}, \mathbf{e}) = 0$ with the condition $(\mathbf{h}, \mathbf{e}) \neq 0$. Note that, unlike the previous cases, bifurcation (3.26) contains the vector $\mathbf{d}$.

4. LOCAL ANALYSIS OF THE STABILITY DOMAIN AT SINGULAR POINTS OF ITS BOUNDARY

Suppose $\mathbf{p} = \mathbf{p}_0$ is a point on the boundary of the parametric resonance domain. We draw a smooth curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ from this point ($\mathbf{p}_0 = \mathbf{p}(0)$, $\varepsilon \geqslant 0$). Some curves will then lie in the stability domain (for small $\varepsilon > 0$), while some will lie in the instability domain. We select the directions of the curves $\mathbf{e} = d\mathbf{p}/d\varepsilon$ (the derivative is taken at $\varepsilon = 0$), which lie in the stability domain. The set of such directions forms a tangential cone to the stability domain at the point on its boundary $\mathbf{p} = \mathbf{p}_0$ [10].

The investigation of the stability along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ is a single-parameter problem. The asymptotic stability is determined by condition $|\rho| < 1$ for all multipliers. By virtue of the continuous dependence on the parameters, multipliers which, for $\mathbf{p} = \mathbf{p}_0$, were inside the unit circle do not leave it for small ε. consequently, the stability along the curve is determined by the behaviour solely of those multipliers which, for $\mathbf{p} = \mathbf{p}_0$, lie on the unit circle. It is obvious that, when there are complex-conjugate multipliers, it is sufficient to investigate the behaviour of only one of these, in view of the symmetry about the real axis.

We will consider different cases of the general position when there are one, two or three parameters in the system.

1. Suppose, at $\mathbf{p} = \mathbf{p}_0$ there is only a pair of simple multipliers $\rho_0 = \exp(i\omega)$ and $\rho_0 = \exp(-i\omega)$, which lie on the unit circle, while the remaining multipliers lie inside the unit circle. Using expression (3.8) for a multiplier perturbed along the curve, we obtain an expression for the absolute value of $|\rho|$ in the form

$$|\rho|^2 = 1 + 2[(\mathbf{r},\mathbf{e})\cos\omega + (\mathbf{k},\mathbf{e})\sin\omega]\varepsilon + o(\varepsilon) \tag{4.1}$$

Then, the necessary condition for the asymptotic stability of the system takes the form

$$(\mathbf{r}\cos\omega + \mathbf{k}\sin\omega, \mathbf{e}) \leqslant 0 \tag{4.2}$$

The behaviour of the simple multipliers $\rho_0 = 1$ and $\rho_0 = -1$ is described by the same relation (4.1) on substituting $\omega = 0$ and $\omega = \pi$ respectively. Hence, we obtain the necessary condition for stability in cases when only these multipliers lie on the unit circle

$$\rho_0 = 1: \quad (\mathbf{r},\mathbf{e}) \leqslant 0; \qquad \rho_0 = -1: \quad (\mathbf{r},\mathbf{e}) \geqslant 0 \tag{4.3}$$

When the non-rigorous inequalities (4.2) and (4.3) are replaced by rigorous inequalities the necessary conditions for stability along the curve become the sufficient conditions.

2. Suppose, at $\mathbf{p} = \mathbf{p}_0$ there is a pair of double multipliers $\rho_0 = \exp(i\omega)$ and $\rho_0 = \exp(-i\omega)$, to which second-order Jordan blocks correspond. Using expression (3.15), which describes the decay of the double ρ_0, we obtain the change in the absolute value of the multipliers in the form

$$|\rho|^2 = 1 + 2\,\mathrm{Re}\sqrt{\varepsilon[(\mathbf{s}_1,\mathbf{e}) + i(\mathbf{s}_2,\mathbf{e})]} + o(\varepsilon^{1/2}) \tag{4.4}$$

$$\mathbf{s}_1 = \mathbf{f}_1\cos 2\omega + \mathbf{f}_2\sin 2\omega, \quad \mathbf{s}_2 = \mathbf{f}_2\cos 2\omega - \mathbf{f}_1\sin 2\omega$$

The square root in (4.4) takes two different values, which differ in sign. Consequently, for stability ($|\rho| < 1$) it is necessary for the second term in (4.4) to be equal to zero. This condition is satisfied if and only if the radicand is real and non-positive. Hence we have

$$(\mathbf{s}_1, \mathbf{e}) \leqslant 0, \qquad (\mathbf{s}_2, \mathbf{e}) = 0 \tag{4.5}$$

Using conditions (4.5) in relation (3.15), we can write

$$|\rho|^2 = 1 + (\mathbf{s}_3 - \mathbf{s}_1, \mathbf{e})\varepsilon + o(\varepsilon), \quad \mathbf{s}_3 = \mathbf{q}_1 \cos\omega + \mathbf{q}_2 \sin\omega$$

Hence, we directly obtain the condition for asymptotic stability

$$(\mathbf{s}_3 - \mathbf{s}_1, \mathbf{e}) \leqslant 0 \tag{4.6}$$

Thus, relations (4.5) and (4.6) are the necessary conditions for asymptotic stability.

The bifurcations of the double multipliers $\rho_0 = -1$ and $\rho_0 = 1$ with second-order Jordan blocks are described by the same relations if we substitute the values $\omega = 0$ and $\omega = \pi$, respectively, and take into account the relations $\mathbf{f}_2 = \mathbf{q}_2 = 0$. Consequently, the conditions for asymptotic stability (4.5) and (4.6) in these cases take the form

$$\rho_0 = 1: \qquad (\mathbf{f}_1, \mathbf{e}) \leqslant 0 \quad (\mathbf{q}_1 - \mathbf{f}_1, \mathbf{e}) \leqslant 0$$

$$\rho_0 = -1: \qquad (\mathbf{f}_1, \mathbf{e}) \leqslant 0 \quad (\mathbf{q}_1 + \mathbf{f}_1, \mathbf{e}) \geqslant 0 \tag{4.7}$$

Conditions (4.5), (4.6) and also (4.7), when the non-rigorous inequalities are replaced by rigorous ones, become the sufficient conditions for asymptotic stability along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$.

3. We will consider the case of a triple multiplier $\rho_0 = 1$ with a third-order Jordan block. The bifurcation of $\rho_0 = 1$ is described by expression (3.20). For a non-zero radicand in (3.2) the multiplier $\rho_0 = 1$ splits into three simple multipliers, corresponding to three different complex values of the cube root. It is obvious that at least one of the multipliers always lies outside the unit circle, which leads to parametric resonance. Consequently, for stability along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ it is necessary for the radicand in (3.20) to be equal to zero, namely

$$(\mathbf{g}, \mathbf{e}) = 0 \tag{4.8}$$

When condition (4.8) is satisfied the bifurcation of the triple multiplier $\rho_0 = 1$ is described by expressions (3.26). From the first relation of (3.26) we obtain an expression for the absolute values of the corresponding multipliers

$$|\rho|^2 = 1 \pm 2\,\mathrm{Re}\sqrt{\varepsilon(\mathbf{h}, \mathbf{e})} + o(\varepsilon^{1/2})$$

Hence we obtain an additional necessary condition for stability

$$(\mathbf{h}, \mathbf{e}) \leqslant 0 \tag{4.9}$$

Taking inequality (4.9) into account, we can write the expression for the absolute values of the multipliers, described by the first relation of (3.26), in the form

$$|\rho|^2 = 1 + \left[(\mathbf{t} - \mathbf{h}, \mathbf{e}) - \frac{(\mathbf{Re}, \mathbf{e}) - (\mathbf{g}, \mathbf{d})}{(\mathbf{h}, \mathbf{e})}\right]\varepsilon + o(\varepsilon), \quad \mathbf{d} = \frac{1}{2}\frac{d^2\mathbf{p}}{d\varepsilon^2} \tag{4.10}$$

For asymptotic stability ($|\rho| < 1$) it is necessary that the expression in square brackets in (4.10) should be non-positive. After elementary algebra, taking (4.9) into account, this inequality can be reduced to the form

$$(\mathbf{g}, \mathbf{d}) \geqslant (\mathbf{Re}, \mathbf{e}) - (\mathbf{t} - \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e}) \tag{4.11}$$

The third multiplier, described by the second relation of (3.26), is real. The necessary condition for stability

$$\frac{(\mathbf{Re},\mathbf{e})-(\mathbf{g},\mathbf{d})}{(\mathbf{h},\mathbf{e})}\leqslant 0$$

for this multiplier, taking (4.9) into account, can be reduced to the form $(\mathbf{g}, \mathbf{d}) \leqslant (\mathbf{Re}, \mathbf{e})$, which, in combination with (4.11), determines the interval to which the quantity $(\mathbf{g}, \mathbf{d})$ belongs,

$$(\mathbf{Re}, \mathbf{e}) - (\mathbf{t} - \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e}) \leqslant (\mathbf{g},\mathbf{d}) \leqslant (\mathbf{Re}, \mathbf{e}) \tag{4.12}$$

In order that the double inequality (4.12) should have solutions for $\mathbf{d}$, it is necessary that $(\mathbf{t} - \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e}) \geqslant 0$ or, taking inequality (4.9) into account,

$$(\mathbf{t} - \mathbf{h}, \mathbf{e}) \leqslant 0 \tag{4.13}$$

Note that expressions (3.26), used above, hold when $(\mathbf{h}, \mathbf{e}) \neq 0$. The case of double degeneracy $(\mathbf{g}, \mathbf{e}) = (\mathbf{h}, \mathbf{e}) = 0$, strictly speaking, requires additional investigation. However, in the case considered we can avoid general discussions. In fact, conditions $(\mathbf{g}, \mathbf{e}) = (\mathbf{h}, \mathbf{e}) = 0$ specify a straight line in three-dimensional parameter space. Moreover, it is well known that the set of directions $\mathbf{e}$ of stable curves (the tangential cone) in the case of a "break of an edge" $D_1(1^3)$ is a plane angle [6]. Consequently, no curve can be drawn in the stability domain along the isolated direction $(\mathbf{g}, \mathbf{e}) = (\mathbf{h}, \mathbf{e}) = 0$, $(\mathbf{t} - \mathbf{h}, \mathbf{e}) > 0$, which violates conditions (4.13).

Thus, relations (4.8), (4.9) and (4.13) are the necessary conditions for stability along the curve. If we add (4.12) to relations (4.8), (4.9) and (4.13) and replace the non-rigorous inequalities by rigorous equalities, we obtain the sufficient conditions for asymptotic stability.

The case of a triple multiplier $\rho_0 = -1$ is similar to that considered above. The corresponding inequalities have the form

$$(\mathbf{g},\mathbf{e})=0 \quad (\mathbf{h},\mathbf{e})\leqslant 0 \quad (\mathbf{t}+\mathbf{h},\mathbf{e})\geqslant 0 \tag{4.14}$$

$$(\mathbf{Re}, \mathbf{e}) \leqslant (\mathbf{g}, \mathbf{d}) \leqslant (\mathbf{Re}, \mathbf{e}) - (\mathbf{t} + \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e}) \tag{4.15}$$

The cases considered exhaust all possible versions encountered in the case of the general position for one, two or three parameters.

If there are several multipliers on the unit circle at $\mathbf{p} = \mathbf{p}_0$, we need to combine the relations which determine the stable perturbations for each of these. The general result can be formulated as follows.

Theorem 2. The tangential cones to the stability domain at points on its boundary of the type (2.1)–(2.3) are defined by the relations

$$\begin{aligned}
K_{B_1} &= \{\mathbf{e}: \ (\mathbf{r},\mathbf{e})\leqslant 0\}\\
K_{B_2} &= \{\mathbf{e}: \ (\mathbf{r},\mathbf{e})\geqslant 0\}\\
K_{B_3} &= \{\mathbf{e}: \ (\mathbf{r}\cos\omega+\mathbf{k}\sin\omega,\mathbf{e})\leqslant 0\}\\
K_{C_1} &= \{\mathbf{e}: \ (\mathbf{f}_1,\mathbf{e})\leqslant 0, \ (\mathbf{q}_1-\mathbf{f}_1,\mathbf{e})\leqslant 0\}\\
K_{C_2} &= \{\mathbf{e}: \ (\mathbf{f}_1,\mathbf{e})\leqslant 0, \ (\mathbf{q}_1+\mathbf{f}_1,\mathbf{e})\geqslant 0\}\\
K_{D_1} &= \{\mathbf{e}: \ (\mathbf{g},\mathbf{e})=0, \ (\mathbf{h},\mathbf{e})\leqslant 0, \ (\mathbf{t}-\mathbf{h},\mathbf{e})\leqslant 0\}\\
K_{D_2} &= \{\mathbf{e}: \ (\mathbf{g},\mathbf{e})=0, \ (\mathbf{h},\mathbf{e})\leqslant 0, \ (\mathbf{t}+\mathbf{h},\mathbf{e})\geqslant 0\}\\
K_{D_3} &= \{\mathbf{e}: \ (\mathbf{s}_2,\mathbf{e})=0, \ (\mathbf{s}_1,\mathbf{e})\leqslant 0, \ (\mathbf{s}_3-\mathbf{s}_1,\mathbf{e})\leqslant 0\}
\end{aligned} \tag{4.16}$$

The tangential cones for combined types (2.4) and (2.5) are obtained by truncating the tangential cones obtained for each of the subtypes. For example, in the case C_1B_2 we have $K_{C_1B_2} = K_{C_1} \cap K_{B_2}$.

In the case D_3 all the continuous curves, issuing in the directions $\mathbf{e} \in K_{D_3}$, which satisfy rigorous inequalities (4.16), lie in the stability domain for fairly small $\varepsilon > 0$, while in the cases D_1 and D_2 only curves which satisfy the conditions on the second derivative $\mathbf{d} = (d^2\mathbf{p}/d\varepsilon^2)/2$ (4.12) and (4.15) respectively, lie in the stability domain. Note that the tangential cones $K_{D_1}, K_{D_2}, K_{D_3}$ are degenerate (are plane angles in three-dimensional parameter space).

Theorem 2 enables one, using information at a point on the boundary of the stability domain (from the first derivatives of the operator **G** from (1.1) with respect to the parameters, and also the values of the multipliers and the corresponding eigenvectors and associated vectors, calculated at $\mathbf{p} = \mathbf{p}_0$), to determine, in the first approximation, the stability domain in the neighbourhood of this point. Relations (4.16) give an explicit representation of the stability domain. For example, in the case of the non-singular point B_1, the tangential cone is determined by the inequality $(\mathbf{r}, \mathbf{e}) \leqslant 0$. Consequently, the tangential plane to the boundary of the stability domain is specified by the equation $(\mathbf{r}, \mathbf{e}) = 0$, while the vector r is normal to the boundary lying in the parametric resonance domain (Fig. 4). In the case of the singular point C_1, the tangential cone is formed by the intersecting half-planes (half-spaces) $(\mathbf{f}_1, \mathbf{e}) \leqslant 0$ and $(\mathbf{q}_1 - \mathbf{f}_1, \mathbf{e}) \leqslant 0$. These inequalities define the plane (dihedral) angle in the space of two (three) parameters, which is the first approximation to the stability domain (Fig. 4).

5. EXAMPLE

Consider the plane vibrations of an articulated pipe conveying fluid (Fig. 5). Parts of the pipe are connected using elastic joints with stiffnesses c_1 and c_2 and have lengths l_1 and l_2, respectively. The right end of the pipe is free. A fluid with a mass per unit length m and a pulsating velocity $u(t) = U(1 + \nu \sin \Omega t)$ flows in the pipe. We will denote the mass per unit length of the pipe by M. The system has two degrees of freedom. We will choose φ and ψ – the angles of deviation of the parts of the tube from the horizontal axis, as the generalized coordinates.

The linearized equations of motion of the system in dimensionless variables have the form [11]

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0 \tag{5.1}$$

$$\mathbf{q} = \begin{Vmatrix} \varphi \\ \psi \end{Vmatrix}, \quad \mathbf{M} = \begin{Vmatrix} \lambda^3 + 3\lambda^2 & 1.5\lambda \\ 1.5\lambda & 1 \end{Vmatrix}, \quad \mathbf{B} = v(\tau)\begin{Vmatrix} \lambda^2 & 2\lambda \\ 0 & 1 \end{Vmatrix}$$

$$\mathbf{C} = \begin{Vmatrix} \sigma + 1 - \lambda f(\tau) & -1 + \lambda f(\tau) \\ -1 & 1 \end{Vmatrix}$$

$$\tau = \alpha t, \quad f(\tau) = \dot{v}(\tau) + \frac{v^2(\tau)}{\mu}, \quad \lambda = \frac{l_1}{l_2}, \quad \mu = \frac{3m}{M + m}, \quad \sigma = \frac{c_1}{c_2}$$

$$v(\tau) = V(1 + \nu \sin w\tau), \quad V = \frac{U\mu}{\alpha l_2}, \quad w = \frac{\Omega}{\alpha}, \quad \alpha^2 = \frac{\mu c_2}{m l_2^3}$$

where the dots denote differentiation with respect to the dimensionless time τ.

The stability of the trivial equilibrium position of the system $\varphi = \psi = 0$ was investigated previously [11] for a number of values of the parameters. In this problem we are interested primarily in singularities of the boundary of the stability domain.

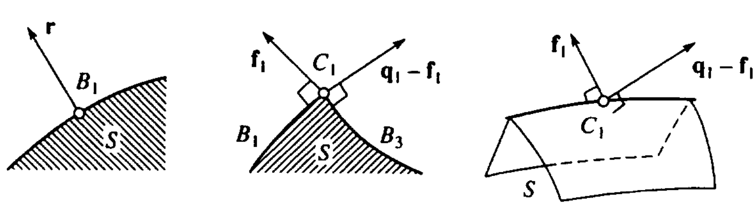

Fig. 4

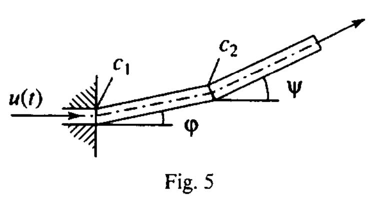

Fig. 5

We will write Eqs (5.1) in the form (1.1)

$$\dot{\mathbf{x}} = \mathbf{G}(\tau)\mathbf{x}, \quad \mathbf{x} = \begin{Vmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Vmatrix}, \quad \mathbf{G}(\tau) = \begin{Vmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{B} \end{Vmatrix} \tag{5.2}$$

where $\mathbf{0}$ is the zero matrix of dimension 2×2, while the matrix operator $\mathbf{G}(\tau)$ with the period $T = 2\pi/w$ depends continuously on the dimensionless parameters λ, μ, σ, w, ν, V.

We will fix the parameters $\lambda = \sigma = \mu = 1$, which corresponds to similar lengths of the parts of the pipe, equal stiffnesses at the joints and a mass per unit length of the fluid half that of the mass per unit length of the pipe. We will investigate the stability of system (5.2) in the three-dimensional parameter space $\mathbf{p} = (w, \nu, V)$, where the quantities w and ν describe the frequency and amplitude of pulsations, while V represents the mean flow velocity. When $\nu = 0$ we have $\upsilon(t) = V = \text{const}$, i.e. a steady system. The critical velocity (the minimum velocity above which the system becomes unstable) in this case is $V_{cr} = (6.2 - 0.4\sqrt{29})^{1/2} \approx 2.0115$.

We will investigate the stabilizing influence of the fluid pulsations at supercritical velocities $V > V_{cr}$. We choose the supercritical value of the velocity $V = 2.8$ and the pulsation frequency $w = 8$. By a numerical analysis we obtain the value of the pulsation amplitude corresponding to a point on the boundary of the stability domain. To do this we need to calculate the monodromy matrix and its multipliers for different ν. As a result we obtain $\nu = 0.737$. The monodromy matrix $\mathbf{F}$ was calculated by integrating Eqs (1.4) from 0 to T using the Runge-Kutta method. The following multipliers correspond to the point $\mathbf{p}_0 = (8; 0.737; 2.8)$ of parameter space

$$\rho_{1,2} = \exp(\pm 0.882i), \qquad \rho_3 = 0.535, \qquad \rho_4 = 0.152$$

The point $\mathbf{p}_0$ on the boundary of the stability domain is regular, since the multipliers $\rho_{1,2}$, lying on the unit circle, are simple.

By calculating the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$, corresponding to the multiplier ρ_1, and the derivatives of the monodromy matrix using Eqs (1.6), we determine the vectors $\mathbf{r}$ and $\mathbf{k}$ from (3.7). The tangential cone K_{B3} to the stability domain at the point $\mathbf{p}_0$, by (4.16), is described by the expression

$$(\mathbf{r}\cos\omega + \mathbf{k}\sin\omega, \mathbf{e}) \leq 0, \quad \omega = \arg\rho_1$$

The vector

$$\mathbf{n} = \mathbf{r}\cos\omega + \mathbf{k}\sin\omega = (0.03; -2.05; 0.51)$$

is the normal to the boundary of the stability domain and lies in the instability domain. The boundary of the stability domain obtained numerically and the vector $\mathbf{n}$ are shown in Fig. 6. This information can be used to stabilize (or destabilize) the system by gradient methods. For example, the use of the formula for the change in the parameters $\delta\mathbf{p} = -\alpha\mathbf{n}$, where $\alpha > 0$ is a small gradient step, leads to a reduction in the modulus of the multiplier.

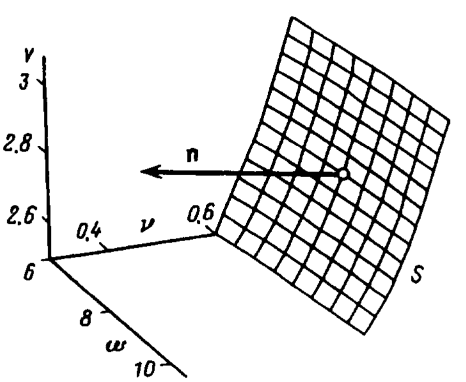

Fig. 6

$$\delta|\rho|^2 = -2\alpha(\mathbf{n},\mathbf{n}) < 0$$

which means stabilization of the system. Gradient procedures can be used for the motion along the boundary of the stability domain. In many cases this leads to arrival at a singularity of the boundary of the stability domain (for example, at an edge).

Consider the point $\mathbf{p}_0 = (3.643; 0.5555; 2.6)$ in parameter space, in which the monodromy matrix $\mathbf{F}_0$ has the following eigenvalues (multipliers)

$$\rho_1 = \rho_2 = -1, \qquad \rho_3 = 0.225, \qquad \rho_4 = 0.026 \tag{5.3}$$

It turns out that the second-order Jordan blocks (3.9) and (3.10) correspond to the double multiplier (we denote it by $\rho_0 = -1$)

$$\mathbf{u}_0 = (0.92; 0.7; -0.59; 2.49)^T, \qquad \mathbf{u}_1 = (0.11; -0.35; 0.12; 1)^T$$
$$\mathbf{v}_0 = (3.34; -2.27; 1.21; -0.3)^T, \qquad \mathbf{v}_1 = (-0.57; 1.11; 0.4; 0.4)^T \tag{5.4}$$

which satisfy normalization conditions (3.11).

It follows from (5.3) that only the double multiplier $\rho_0 = -1$ pertains to the boundary, while the other two lie inside the unit circle. Hence, the point ρ_0 lies on the boundary of the stability domain in the parameter space $\mathbf{p} = (w, \nu, V)$. In the neighbourhood of the point $\mathbf{p}_0$ we reconstruct the stability domain using the results of Sections 2 and 4. According to notation (2.2), the type C_2 singularity – a "dihedral angle", corresponds to the point $\mathbf{p}_0$. We will determine the tangential cone to the stability domain at this point. Using the vectors (5.4) and calculating the first derivatives of the monodromy matrix with respect to the parameters using (1.6), we obtain from (3.14)

$$\mathbf{f}_1 = (-5.15; 45.2; -7.77), \qquad \mathbf{q}_1 = (4.49; -31.1; 3.16) \tag{5.5}$$

By Theorem 2 these vectors define the tangential cone to the stability domain at the point $\mathbf{p}_0$

$$K_{C_2} = \{\mathbf{e}: \ (\mathbf{f}_1, \mathbf{e}) \leq 0, \ \ (\mathbf{q}_1 + \mathbf{f}_1, \mathbf{e}) \geq 0\} \tag{5.6}$$

The vectors $\mathbf{f}_1$ and $-(\mathbf{q}_1 + \mathbf{f}_1)$ are normals to the sides of the "dihedral angle" lying in the parametric resonance domain. The vector, tangential to the edge of the "dihedral angle", is equal to

$$\mathbf{e}_\tau = (\mathbf{q}_1 + \mathbf{f}_1) \times \mathbf{f}_1 = (98.8; 18.6; 42.7)$$

In Fig. 7, on the left, we show the dihedral angle (5.6), which is an approximation of the boundary of the stability domain in the neighbourhood of the point $\mathbf{p}_0$. For comparison we show in Fig. 7, on the right, the boundary of the stability domain obtained numerically, which confirms the presence of a singularity and agrees well with the results obtained. Note that, to obtain an approximation of the stability domain, only a single integration with respect to time of differential equations (1.4) from 0 to T is necessary to find the monodromy matrix and to evaluate the three integrals (1.6). Information on the tangential cone (5.6) can be used for further motion along the edge of the boundary of the stability domain in order to optimize the system stability.

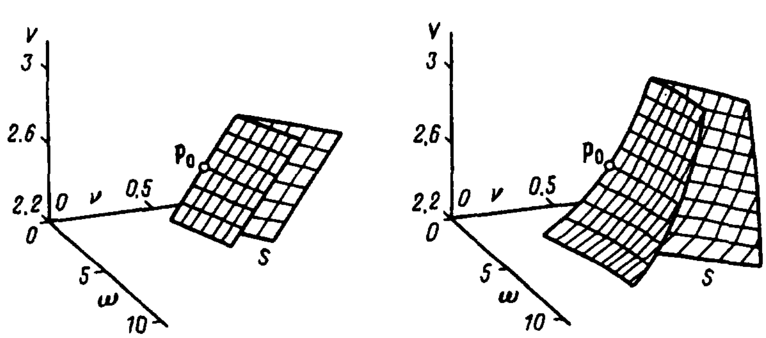

Fig. 7

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